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Solvable separable nonlocal potential models

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Abstract. We introduce a generalized Yamaguchi rank-one separable potential with *three* parameters which has the separable Yukawa or central Yamaguchi potentials as limiting cases. The three parameters can be obtained, when possible, from the set of experimental data (binding energy, scattering length and effective range) by solving a simple cubic equation. We also discuss a rank-two separable potential with one of the strengths infinite. For both potentials the solutions can be expressed analytically.

1. Introduction

Separable nonlocal potentials have proved useful in the study of the few-nucleon problems, particularly in the determination of the three-body binding energy using a two-body separable interaction. Such potentials also yield simple analytical solutions in two-body systems, which is helpful for gaining physical insight into these systems.

In this paper, we look at some novel separable interactions which we apply to the triplet neutron–proton interaction for small positive energies together with the negative energy corresponding to the (deuteron) binding energy. In sections 2 and 3 we consider particular rank-one and rank-two potentials, respectively. As an application of the analytic solutions, we show explicitly that the effective range has very little model dependence once the binding energy and scattering length are fixed.

2. Generalized Yamaguchi rank-one separable potential

We introduce a separable potential which interpolates between a separable Yukawa and a central Yamaguchi potential [1]. It is useful because it can give any value of the effective range between those for these two potentials. The nonlocal potential is written in the form

$$V(p, p') = -\frac{2}{\pi}\lambda\Lambda_0 pp' f(p) f(p') \tag{2.1}$$

with the form factor

$$f(p) = \frac{\beta_1\beta_2}{\sqrt{\beta_1^2 + p^2}\sqrt{\beta_2^2 + p^2}}. \tag{2.2}$$

The quantity λ is a dimensionless parameter which is set equal to one when the potential supports a zero-energy bound state. Thus Λ_0 is found from the zero-energy case and is given by

$$\Lambda_0 = \frac{\beta_1\beta_2}{\beta_1+\beta_2}. \tag{2.3}$$

At the deuteron bound-state energy for which $\alpha \neq 0$,

$$\lambda = \frac{(\alpha+\beta_1)(\alpha+\beta_2)}{\beta_1\beta_2}. \tag{2.4}$$

The effective range formula for the nonlocal potential is

$$k\cot\delta = \frac{\beta_1\beta_2}{\beta_1+\beta_2}\left[\frac{1}{\lambda}\left(1+\frac{k^2}{\beta_1^2}\right)\left(1+\frac{k^2}{\beta_2^2}\right) - \left(1-\frac{k^2}{\beta_1\beta_2}\right)\right] \tag{2.5}$$

$$= -\frac{1}{a_t} + \frac{1}{2}r_0k^2 - Pr_0^3k^4. \tag{2.6}$$

Thus, the scattering length, effective range and shape parameter P are given by the expressions

$$a_t = \frac{\lambda(\beta_1+\beta_2)}{\alpha(\alpha+\beta_1+\beta_2)} \tag{2.7}$$

$$r_0 = \frac{2\beta_1\beta_2}{\beta_1+\beta_2}\left[\frac{1}{\lambda}\left(\frac{1}{\beta_1^2}+\frac{1}{\beta_2^2}\right)+\frac{1}{\beta_1\beta_2}\right] \tag{2.8}$$

and

$$Pr_0^3 = -\frac{1}{\lambda\beta_1\beta_2(\beta_1+\beta_2)} = \frac{a_t\alpha(2-\alpha r_0)-2}{2a_t\alpha^4}. \tag{2.9}$$

For $\beta_1 = \beta_2$, we recover the Yamaguchi form factor, whereas in the limit as β_2 approaches infinity while β_1 remains finite, we obtain the Yukawa form factor (see, for example, [2]).

Let us consider the Yamaguchi case in more detail. We introduce a dimensionless quantity $x = \beta_1/\alpha$. Then the strength parameter is given by

$$\lambda = \frac{(x+1)^2}{x^2} \tag{2.10}$$

and the triplet scattering length and effective range are, respectively,

$$a_t = \frac{2(x+1)^2}{\alpha x(2x+1)} \tag{2.11}$$

and

$$r_0 = \frac{1}{\alpha}\left(\frac{2x}{(x+1)^2}+\frac{1}{x}\right). \tag{2.12}$$

The following equation holds for the Yamaguchi potential:

$$\frac{\alpha r_0}{2} - \frac{a_t\alpha-1}{a_t\alpha} = \frac{1}{2x(x+1)^2}. \tag{2.13}$$

Now for weak binding, such as we have in the deuteron, α is small and x is large. Thus the quantity on the right-hand side of equation (2.13) is small. Consequently the effective range is determined mainly by the scattering length and deuteron binding energy.

For the Yamaguchi potential, using neutron–proton data, namely, $\alpha = 0.2316\ \mathrm{fm}^{-1}$ and $a_t = 5.419$ fm, we obtain

$$\tfrac{1}{2}\alpha r_0 = 0.2032 + 0.0017 \tag{2.14}$$

and $r_0 = 1.754 + 0.015 = 1.769$ fm.

We can extend this result to the case of a generalized Yamaguchi potential. First of all we will show that given α and a_t, r_0 can have only a small range of values with 1.754 and 1.769 fm as the lower and upper limit, respectively. Defining ξ as

$$\xi = \frac{\alpha r_0}{2} - \frac{a_t\alpha - 1}{a_t\alpha} \tag{2.15}$$

we find that ξ can be written as

$$\xi = \frac{\alpha^3}{(\beta_1+\beta_2)(\alpha+\beta_1)(\alpha+\beta_2)} = \frac{1}{(x+y)(1+x)(1+y)} \tag{2.16}$$

where $y = \beta_2/\alpha$.

We can obtain the values of β_1 and β_2, or equivalently of x and y, as solutions of a cubic equation. Let us define the quantities p and q by the equations

$$\xi = \frac{1}{p} \qquad \text{and} \qquad \alpha a_t = \frac{p}{q}. \tag{2.17}$$

Then

$$p = (1+x)(1+y)(x+y) \tag{2.18}$$

$$q = xy(1+x+y). \tag{2.19}$$

As can be seen in equations (2.18) and (2.19), p and q are functions of both x and y, but we can eliminate y to obtain an equation involving only x, i.e.

$$\frac{p}{1+x} - \frac{q}{x} = x \tag{2.20}$$

which can be written as a cubic equation for x

$$x^3 + x^2 + (p-q)x + q = 0. \tag{2.21}$$

There is an identical equation for y. If equation (2.21) has two positive roots, one gives the required value of x and the other the value of y. If we now take the values of α and a_t used earlier and the 'experimental' value $r_0 = 1.752$ fm [3], we find that equation (2.21) does not have a pair of positive solutions. However, if the value of r_0 is changed to 1.756 fm, there are two positive solutions, namely, 3.97 and 36.59. Table 1 shows the range of values for r_0, with fixed α and a_t, covered by the generalized Yamaguchi separable potential. Note that this range is small with the Yamaguchi and Yukawa values as extremes.

Table 1. The roots x and y of the cubic equation with $a_t = 5.419$ fm for various values of the effective range r_0. The asterisk (*) indicates that there is no pair of positive solutions.

r_0	<1.754	1.7549	1.756	1.760	1.764	1.768	1.76934	>1.77
x	*	3.921	3.972	4.187	4.501	5.126	6.03	*
y	*	∞	36.592	15.475	10.471	7.498	6.03	*
		Yukawa					Yamaguchi	

In order to study further the model dependence of r_0 and to simplify the expressions, we define a dimensionless quantity γ such that

$$\gamma = \frac{\beta_1\beta_2}{\beta_1^2 + \beta_1\beta_2 + \beta_2^2} = \frac{xy}{x^2 + xy + y^2}. \tag{2.22}$$

The quantity γ is zero for the separable Yukawa potential and $\gamma = \frac{1}{3}$ for the Yamaguchi potential. For the values of β_1 and β_2 for the case mentioned above when $r_0 = 1.756$ fm, we have $\gamma = 0.0969$. If we rewrite equation (2.15) as

$$\frac{\alpha r_0}{2} = \frac{\alpha a_t - 1}{\alpha a_t} + \xi \tag{2.23}$$

it is clear that the model dependence of r_0 is contained in the ξ term. Expanding ξ in terms of $\epsilon = a_t\alpha - 1$ ($\epsilon = 0.2550$ for the neutron–proton case), we obtain

$$\xi = c_0 + c_1\epsilon + c_2\epsilon^2 + c_3\epsilon^3 + \cdots \tag{2.24}$$

and we find that $c_0 = c_1 = c_2 = 0$ and $c_3, c_4, \ldots,$ are functions of γ. Explicit expressions for c_3 and c_4 are $c_3 = \gamma^2(1+\gamma)$ and $c_4 = -\gamma^2(1 + 2\gamma + 4\gamma^2 + 3\gamma^3)$. In particular, as γ varies from 0 to $\frac{1}{3}$, c_3 increases monotonically from 0 to $\frac{4}{27}$.

3. Rank-two separable potential

3.1. General results

A rank-two separable potential with Yamaguchi form factors leads to algebraic expressions for most quantities of interest. In the notation of [2], the potential has the form

$$V(p, p') = \frac{2}{\pi} pp'[\lambda_1 f_1(p) f_1(p') + \lambda_2 f_2(p) f_2(p')]. \tag{3.1}$$

The scattering phase shifts and the deuteron energy can be written in terms of the integrals

$$I_{ij}(k^2) = \frac{2}{\pi}\int_0^\infty \frac{f_i(q) f_j(q)}{k^2 - q^2} q^2\,\mathrm{d}q \qquad i, j = 1, 2 \tag{3.2}$$

where $I_{ij}(k^2)$ is the principal value of the integral when $k^2 > 0$. The scattering phase shift $\delta(k)$ is given by

$$k\cot\delta(k) = -\frac{\mathcal{D}(k^2)}{\mathcal{N}(k)} \tag{3.3}$$

where

$$\mathcal{D}(k^2) = [1 - \lambda_1 I_{11}(k^2)][1 - \lambda_2 I_{22}(k^2)] - \lambda_1\lambda_2 I_{12}^2(k^2) \tag{3.4}$$

and

$$\mathcal{N}(k) = \lambda_1 f_1^2(k)[1 - \lambda_2 I_{22}(k^2)] + 2\lambda_1\lambda_2 f_1(k) f_2(k) I_{12}(k^2) + \lambda_2 f_2^2(k)[1 - \lambda_1 I_{11}(k^2)]. \tag{3.5}$$

The bound-state energy $-\alpha^2 < 0$ can be obtained by solving the equation

$$\mathcal{D}(-\alpha^2) = 0. \tag{3.6}$$

The bound-state wavefunction in momentum space is

$$\tilde{u}(p) = N\frac{p}{p^2 + \alpha^2}\left[\lambda_1 f_1(p) + \frac{1 - \lambda_1 I_{11}(-\alpha^2)}{I_{12}(-\alpha^2)} f_2(p)\right] \tag{3.7}$$

where N is the normalization constant.

3.2. Strength parameter λ_2 infinite

We now consider the special case with Yamaguchi form factors,

$$f_1(p) = \frac{\beta_1^2}{p^2+\beta_1^2} \qquad f_2(p) = \frac{\beta_2^2}{p^2+\beta_2^2}. \tag{3.8}$$

Of special interest is the situation for which one of the strength parameters is finite and negative and the other is infinite, simulating an infinite short-range repulsion (repulsive core) when the range parameter associated with the repulsive term is the larger of the two. We set

$$\lambda_1 = -\lambda\Lambda_0 \qquad \lambda_2 = \lambda\Lambda_0\eta \tag{3.9}$$

where eventually η approaches infinity. As with the single-term separable potential of section 2, λ is a dimensionless parameter chosen so that it is unity when the potential supports a zero-energy bound state. Thus

$$\Lambda_0 = \frac{2}{\beta_1}\left(\frac{\beta_1+\beta_2}{\beta_1-\beta_2}\right)^2 \qquad \lambda = \frac{(\beta_1+\alpha)^2}{\beta_1^2}. \tag{3.10}$$

Again setting $\beta_1 = x\alpha$ and $\beta_2 = y\alpha$, we obtain for the scattering length

$$\begin{aligned} a_t &= 2\,\frac{2\beta_1^2\alpha + \beta_1^2\beta_2 + \beta_1\alpha^2 + 2\beta_1\beta_2\alpha + \beta_2\alpha^2}{\beta_1\beta_2\alpha(2\beta_1+\alpha)} \\ &= \frac{2}{\alpha}\frac{2x^2+x^2y+x+2xy+y}{xy(2x+1)} \end{aligned} \tag{3.11}$$

and for the effective range

$$\begin{aligned} r_0 = {} & \big(8x^5y + 4x^5y^2 + 4x^5 + 3x^4y^3 + 4x^4 + 20x^4y + 16x^4y^2 + 16x^3y + x^3 + 8x^3y^3 \\ & + 24x^3y^2 + 8x^2y^3 + 4yx^2 + 16x^2y^2 + 4xy^2 + 4xy^3 + y^3\big) \\ & \times\big(\alpha yx(yx^2 + 2x^2 + 2xy + x + y)^2\big)^{-1}. \end{aligned} \tag{3.12}$$

If we suppose that $\alpha = 0.2316\ \mathrm{fm}^{-1}$, $a_t = 5.419$ fm and r_0 some given value, we can solve for β_1 and β_2. As with the rank-one separable potential, there is a small range of values of r_0 ($r_0 = 1.770$–1.782 fm) for which physical solutions for the β's exist. That the effective range is close to the experimental value of r_0, but does not include it, can be attributed to the simplicity of the model. Again we can determine the model-dependent contribution to the effective range by expanding ξ in terms of ϵ as in equation (2.24). As before we find that $c_0 = c_1 = c_2 = 0$ and c_3 is a simple expression of $\nu = \beta_1/\beta_2$, i.e.

$$c_3 = \frac{4(\nu+1)(4\nu^2+8\nu+1)}{(4\nu+3)^3}. \tag{3.13}$$

As ν varies from 0 to 1 (i.e. $\beta_2 > \beta_1$), c_3 increases monotonically from $\frac{4}{27}$ to 0.303. Although the model dependence of the effective range is approximately the same magnitude as that of the generalized Yamaguchi potential, this potential yields effective ranges which are equal to or larger than the maximum of the generalized Yamaguchi potential. The model dependence of the effective range is also discussed in [4] in which the authors consider the difference between the mixed effective range and the effective range at the deuteron pole. Their result is of the same order of magnitude as ours.

The bound-state wavefunction in coordinate space is given by

$$u(r) = N\left[\mathrm{e}^{-\alpha r} - \frac{(\beta_1+\beta_2)(\beta_2-\alpha)}{(\beta_2-\beta_1)(\beta_2+\alpha)}\mathrm{e}^{-\beta_1 r} + \frac{2\beta_2(\beta_1-\alpha)}{(\beta_2-\beta_1)(\beta_2+\alpha)}\mathrm{e}^{-\beta_2 r}\right]. \tag{3.14}$$

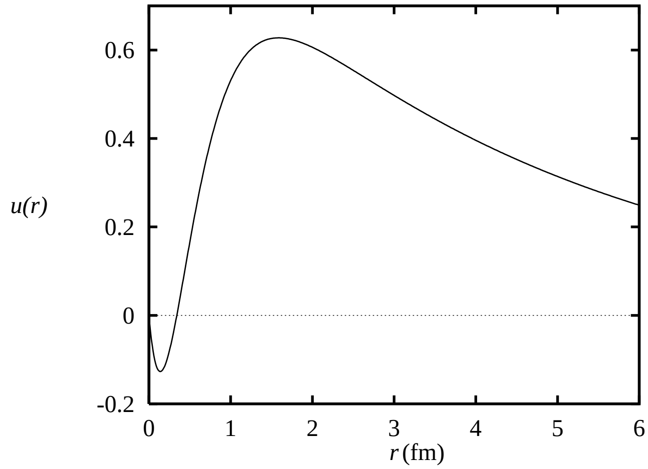

Figure 1. The bound-state wavefunction $u(r)$ of the potential discussed in section 3.2 which gives $r_0 = 1.782$ fm ($x = 11.34$ and $y = 16.55$).

It is interesting to note, as shown in figure 1, that the wavefunction has a node, even though it represents the ground state. Such nodes in the scattering and bound-state wavefunction of nonlocal potentials have been discussed in the literature [5]. More recently nodes in the ground-state wavefunction have been related to supersymmetric potentials in which a deep bound state has been removed by making a supersymmetric transformation which leaves the other bound-state energy and positive-energy phase shifts unchanged [6–8]. To understand the node in the present case, consider the Schrödinger equation in coordinate space. We have obtained a finite non-zero solution even though λ_2 is infinite, which implies that the corresponding potential term must be finite. Consequently, $u(r)$ must be orthogonal to $f_2(r)$, and indeed the wavefunction given above is easily checked to be orthogonal to $e^{-\beta_2 r}$.

4. Discussion

We have introduced a novel rank-one separable potential which interpolates between the separable Yukawa and Yamaguchi potentials. It is useful for analytically discussing the low-energy behaviour of the scattering parameters.

The rank-two separable potential with infinite repulsion yields straightforward analytic expressions for the scattering length and effective range. We find that the deuteron wavefunction has a node, which is due to the orthogonality of the bound-state wavefunction and the form factor of the infinite term of the potential.

In both models we see analytically the small model dependence of the effective range once the binding energy and scattering length are fixed. Although the small model dependence is known from fitting realistic nucleon–nucleon potentials, this feature is elucidated through these simple models.

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